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Problems of the partial stability and detectability of dynamical systems☆

C.A. Alekseyeva, V.I. Vorotnikov, V.A. Feofanova

Nizhnii Tagil, Russia

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Abstract

The conditions under which uniform stability (uniform asymptotic stability) with respect to a part of the variables of the zero equilibrium position of a non-linear non-stationary system of ordinary differential equations signifies uniform stability (uniform asymptotic stability) of this equilibrium position with respect the other, larger part of the variables, which include an additional group of coordinates of the phase vector, are established. These conditions include the condition for uniform asymptotic stability of the zero equilibrium position of the "reduced" subsystem of the original system with respect to the additional group of variables. Since within the conditions obtained the stability with respect to the remaining unmeasured coordinates of the phase vector remains undetermined or is investigated additionally, partial zero-detectability of the original system occurs in this case, and the conditions obtained supplement the series of known results from partial stability theory. The application of the results obtained to problems of the partial stabilization of non-linear controlled systems, particularly to the problem of stabilizing an asymmetric rigid body relative to an assigned direction in an inertial space, is considered. The partial detectability of linear systems with constant coefficients is also investigated.

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The systematic investigation of the problem of stability with respect to a part of the variables posed by Lyapunov[1] was begun in the paper by Rumyantsev,[2] and this problem has undergone definite development.[3–20] In the context of these studies, problems of the detectability and partial detectability of dynamical systems are also of great importance. In general terms, *detectability* of dynamical systems signifies[21–23] that their stability with respect to a part of the variables (the "output") actually leads to stability with respect to all the variables. For linear systems, this problem is the classical and thoroughly studied problem of mathematical control theory. For non-linear systems the detectability problem is considerably more complicated, and a general approach to its investigation began to take shape only during the last 10 years. Nevertheless, in this area there are already some results that were obtained (without loss of generality) as applied to the problem of the detectability of the zero equilibrium position of non-linear dynamical systems. Such a problem is called the zero-detectability problem. A more general problem related to the study of the *partial detectability* of dynamical systems subsequently appeared. In this case, stability with respect to a part of the variables (the "measurable output") signifies stability not with respect to all the variables, but with respect to another, larger part of the variables (the "assigned output" or the V function of the phase variables). The stability with respect to the remaining group of

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E-mail address: vorot@ntiustu.ru (C.A. Alekseyeva).

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variables remains undetermined or requires additional investigation. Only the first steps[24–27] have been taken to solve this problem. As in the case of detectability, partial zero-detectability is analysed without loss of generality.

This paper examines non-linear non-stationary dynamical systems of a general type, for which new conditions for partial zero-detectability are obtained, i.e., the conditions under which uniform stability (uniform asymptotic stability) of the zero equilibrium position of a non-linear non-stationary system of ordinary differential equations with respect to a part of the variables signifies uniform stability (uniform asymptotic stability) of this equilibrium position with respect to another larger part of the variables are obtained. Unlike the existing results,[24–27] the approach used here allows us to aim for a constructive analysis of the structural forms of partially detectable non-linear non-stationary systems, and the results obtained will supplement existing results from partial stability theory. The partial detectability of linear systems with constant coefficients is also investigated.

1. Statement of the problem

Suppose we have a non-linear non-stationary finite-dimensional system of ordinary differential equations (in vector form)

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, \mathbf{0}) \equiv \mathbf{0} \tag{1.1}$$

We separate the variables appearing in the phase vector $\mathbf{x}$ into three parts and represent it in the form (T denotes transposition)

$$\mathbf{x} = (\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{z}^T)^T, \quad \mathbf{y}_1 \in R^m, \quad \mathbf{y}_2 \in R^k, \quad \mathbf{z} \in R^p, \quad m + k + p = n$$

Then system (1.1) consists of the three groups of equations

$$\dot{\mathbf{y}}_i = \mathbf{Y}_i(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}), \quad i = 1, 2, \quad \dot{\mathbf{z}} = \mathbf{Z}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) \tag{1.2}$$

We put $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$ and adopt the standard assumptions for the theory of partial stability ($\mathbf{y}$-stability)[2–20] with regard to the continuity of $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Z}$ in the region

$$t \geq 0, \quad \|\mathbf{y}\| \leq h, \quad \|\mathbf{z}\| < \infty \tag{1.3}$$

as well as with regard to the uniqueness and $\mathbf{z}$-extendibility of the solutions of system (1.2). We will use $\mathbf{x}(t) = \mathbf{x}(t; t_0, \mathbf{x}_0)$ to denote the solution of system (1.2) that satisfies the initial condition $\mathbf{x}_0 = \mathbf{x}(t_0; t_0, \mathbf{x}_0)$.

Definitions. The equilibrium position

$$\mathbf{x} = (\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{z}^T)^T = \mathbf{0} \tag{1.4}$$

of the system of differential Eq. (1.2) is

1) *uniformly* $\mathbf{y}_1$-*stable* (*uniformly* $\mathbf{y}$-*stable*) if for any $\varepsilon > 0$, $t_0 \geq 0$ a $\delta(\varepsilon) > 0$ exists such that the condition $||\mathbf{x}_0|| \leq \delta$ leads to the condition $||\mathbf{y}_1(t; t_0, \mathbf{x}_0)|| < \varepsilon$ ($||\mathbf{y}(t; t_0, \mathbf{x}_0)|| < \varepsilon$) for all $t \geq t_0$;
2) *uniformly asymptotically* $\mathbf{y}_1$-*stable* (*uniformly asymptotically* $\mathbf{y}$-*stable*) if it is uniformly $\mathbf{y}_1$-stable (uniformly $\mathbf{y}$-stable) and a $\Delta > 0$ exists such that for each solution $\mathbf{x}(t; t_0, \mathbf{x}_0)$ of the system (1.2) for which $||\mathbf{x}_0|| < \Delta$, the relation $\lim||\mathbf{y}_1(t; t_0, \mathbf{x}_0)|| \to 0$, $t \to \infty$ holds uniformly with respect to t_0, $\mathbf{x}_0$ from the region $t_0 \geq 0$, $||\mathbf{x}_0|| < \Delta$.

Problem 1. It is required to find the conditions under which uniform $\mathbf{y}_1$ stability (uniform asymptotic $\mathbf{y}_1$-stability) of equilibrium position (1.4) of system (1.2) signifies uniform $\mathbf{y}$-stability (uniform asymptotic $\mathbf{y}$-stability) of this equilibrium position.

2. The conditions for partial detectability of nonlinear systems

From $\mathbf{Y}_2$ we separate the terms that depend only on t and $\mathbf{y}_2$, and we represent $\mathbf{Y}_2$ in the form

$$\mathbf{Y}_2(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) = \mathbf{Y}_2^0(t, \mathbf{y}_2) + \mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z})$$

$$\mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) = \mathbf{Y}_2(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) - \mathbf{Y}_2^0(t, \mathbf{y}_2), \quad \mathbf{R}(t, \mathbf{0}, \mathbf{0}, \mathbf{0}) \equiv \mathbf{R}(t, \mathbf{0}, \mathbf{y}_2, \mathbf{0}) \equiv \mathbf{0}$$

Theorem 1. *Let the following conditions hold*:

a) *the vector function* $\mathbf{Y}_2^0(t, \mathbf{y}_2)$ *and its partial derivatives with respect to* $\mathbf{y}_2$ *are confined to the region* $t \geq 0$, $||\mathbf{y}_2|| \leq h$;
b) *in the region* (1.3) *a continuous vector function* $\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2)$, $\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2) \equiv 0$ *exists such that*

$$\|\mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z})\| \leq \|\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2)\| \tag{2.1}$$

c) *the equilibrium position* $\mathbf{y}_2 = \mathbf{0}$ *of the "reduced" subsystem*

$$\dot{\mathbf{y}}_2 = \mathbf{Y}_2^0(t, \mathbf{y}_2) \tag{2.2}$$

is uniformly asymptotically stable.

Then, if the equilibrium position (1.4) of system (1.2) is uniformly (uniformly asymptotically) $\mathbf{y}_1$-stable, it is uniformly (uniformly asymptotically) $\mathbf{y}$-stable, and $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$.

Proof. When the conditions of the theorem hold for system (2.2), a Lyapunov function $V(t, \mathbf{y}_2)$ exists[6] that is defined and continuous in the region $t \geq 0$, $||\mathbf{y}_2|| \leq h$, has partial derivatives with respect to $\mathbf{y}_2$ that are confined to that region, and satisfies the conditions (the $a_i(r)$ are continuous monotonically increasing functions when $r \in [0,h]$, and $a_i(0) = 0$)

$$a_1(\|\mathbf{y}_2\|) \leq V(t, \mathbf{y}_2) \leq a_2(\|\mathbf{y}_2\|), \quad \dot{V}_{(2.2)}(t, \mathbf{y}_2) \leq -a_3(\|\mathbf{y}_2\|) \tag{2.3}$$

By virtue of systems (1.2) and (2.2), the derivatives of $V(t, \mathbf{y}_2)$ are related by the expression

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2) = \dot{V}_{(2.2)}(t, \mathbf{y}_2) + (\partial V(t, \mathbf{y}_2)/\partial \mathbf{y}_2)\mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) \tag{2.4}$$

which, according to property b and inequality (2.3), can take the form

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2) \leq -a_3(\|\mathbf{y}_2\|) + N\|\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2)\|, \quad N = \text{const} > 0 \tag{2.5}$$

2.1. Uniform stability

Let us assume that the equilibrium position (1.4) of system (1.2) is uniformly $\mathbf{y}_1$-stable. In this case, for any $\varepsilon > 0$, $t_0 \geq 0$ a $\delta(\varepsilon) > 0$ exists such that the condition $||\mathbf{x}_0|| < \delta$ leads to the condition $||\mathbf{y}_1(t; t_0, \mathbf{x}_0)|| < \varepsilon$ for all $t \geq t_0$. Based on inequality (2.3), from inequality (2.5) we obtain

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2) \leq -a_3(a_2^{-1}(V(t, \mathbf{y}_2))) + N\|\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2)\| \tag{2.6}$$

We set

$$\delta_1(\varepsilon) = N^{-1}a_3(a_2^{-1}(a_1(\varepsilon)))$$

We say that $\delta_2(\varepsilon) > 0$ is such that the condition $||\mathbf{y}_1|| < \delta_2$ leads to the condition $||\mathbf{Y}_2^*(\mathbf{y}_1, \mathbf{y}_2)|| < \delta_1$ for $||\mathbf{y}_2|| < \varepsilon$. On the other hand, by virtue of the uniform $\mathbf{y}_1$-stability of the equilibrium position (1.4) of system (1.2), $||\mathbf{y}_1(t; t_0, \mathbf{x}_0)|| < \delta_2(\varepsilon)$ for all $t \geq t_0$ if $||\mathbf{x}_0|| < \delta[\delta_2(\varepsilon)]$. Since the condition $||\mathbf{Y}_2^*(\mathbf{y}_1(t; t_0, \mathbf{x}_0), \mathbf{y}_2)|| < \delta_1$ holds in the region $t \geq 0$, $||\mathbf{y}_2|| < \varepsilon$ for $||\mathbf{x}_0|| < \delta[\delta_2(\varepsilon)]$, it follows from inequality (2.6) that

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2(t; t_0, \mathbf{x}_0)) < 0) \text{ при } V(t, \mathbf{y}_2(t; t_0, \mathbf{x}_0)) = a_1(\varepsilon) \tag{2.7}$$

Let

$$\delta^*(\varepsilon) = \min\{\delta(\varepsilon), \delta(\delta_2(\varepsilon), \delta_3(\varepsilon))\}, \quad \delta_3(\varepsilon) = a_2^{-1}(a_1(\varepsilon))$$

Let us consider an arbitrary solution $\mathbf{x}(t;\, t_0,\, \mathbf{x}_0)$ of system (1.2) with $t_0 \geq 0$, $||\mathbf{x}_0|| < \delta^*(\varepsilon)$. By virtue of inequality (2.3), we have $V(t_0, \mathbf{y}_{20}) < a_2(\delta_3)$ for $||\mathbf{x}_0|| < \delta^*(\varepsilon)$ and, consequently, $V(t_0, \mathbf{y}_{20}) < a_1(\varepsilon)$.We will show that

$$V(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) < a_1(\varepsilon) \text{ при всех } t \geq t_0 \tag{2.8}$$

We will assume, on the contrary, that $V(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) < a_1(\varepsilon)$ when $t \in [t_0, t_1)$, but $V(t_1, \mathbf{y}_2(t_1;\, t_0, \mathbf{x}_0)) = a_1(\varepsilon)$. Then, we clearly have $\dot{V}_{(1.2)}(t_1, \mathbf{y}_2(t_1;\, t_0, \mathbf{x}_0)) \geq 0$, which contradicts condition (2.7).

Based on the condition $V(t, \mathbf{y}_2) \geq a_1(||\mathbf{y}_2||)$, from inequality (2.8) we conclude that $||\mathbf{y}_2(t;\, t_0,\, \mathbf{x}_0)|| < \varepsilon$ for all $t \geq t_0$ if $||\mathbf{x}_0|| < \delta^*(\varepsilon)$.

2.2. Uniform asymptotic stability

We will assume that the equilibrium position (1.4) of system (1.2) is uniformly asymptotically $\mathbf{y}_1$-stable. The uniform $\mathbf{y}_2$-stability of this equilibrium position follows from the part of the theorem that is devoted to uniform stability: for any $\varepsilon > 0$, $t_0 \geq 0$ a $\delta^*(\varepsilon) > 0$ exists such that the condition $||\mathbf{x}_0|| \leq \delta^*$ leads to the condition $||\mathbf{y}_2(t;\, t_0,\, \mathbf{x}_0)|| < \varepsilon$ for all $t \geq t_0$.

We will show that the equilibrium position (1.4) of system (1.2) is also uniformly $\mathbf{y}_2$-attractive. This means that for an assigned $\delta^*(\varepsilon) > 0$ and any $\eta \in (0, \delta^*)$ a number $T(\eta) > 0$ exists such that the condition $t_0 \geq 0$, $||\mathbf{x}_0|| \leq \delta^*$ leads to the condition $||\mathbf{y}_2(t;\, t_0,\, \mathbf{x}_0)|| < \eta$ for all $t \geq t_0 + T(\eta)$.

Under the conditions of the part of the theorem devoted to uniform asymptotic stability, for system (1.2) a Lyapunov function $V(t, \mathbf{y}_2)$ exists that satisfies not only conditions (2.3), but also equality (2.4), in which the relation

$$\|\mathbf{R}(\mathbf{y}_1(t;\, t_0, \mathbf{x}_0), \mathbf{y}_2, \mathbf{z})\| \Rightarrow 0, \quad t \to \infty \tag{2.9}$$

holds uniformly over $\lambda \leq ||\mathbf{y}_2|| \leq \mu$, $||\mathbf{z}|| < \infty$ and $t_0 \geq 0$, $||\mathbf{x}_0|| < \Delta < \delta^*$, where $\Delta > 0$ defines the region of uniform $\mathbf{y}_1$-attraction of the equilibrium position (1.4) of (1.2). Let $\eta \in (0, \Delta)$ be given. By virtue of conditions (2.4), (2.9) a $T_1(\eta) > 0$ exists such that for

$$t \geq T_1(\eta), \quad a_2^{-1}(a_1(\eta)) \leq \|\mathbf{y}_2\| \leq \varepsilon \;(\eta < \delta^*(\varepsilon) < a_2^{-1}(a_1(\varepsilon)) < \varepsilon), \quad \|\mathbf{x}_0\| < \Delta < \delta^*$$

the inequality

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) \leq -a_3(a_2^{-1}(a_1(\eta)))/2 \tag{2.10}$$

and, consequently, the inequality

$$\dot{V}_{(1.2)}(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) < 0 \text{ при } V(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) = a_1(\eta) \tag{2.11}$$

hold if $t \geq T_1(\eta)$. We set

$$t_0' = \max[t_0, T_1(\eta)], \quad T_2(\eta) = a_3^{-1}(a_2^{-1}(a_1(\eta)))[2a_2(\eta) - a_1(\eta)]$$

We will show that a time $t_* \in (t'_0,\, t'_0 + T_2(\eta))$ exists, for which

$$V(t_*, \mathbf{y}_2(t_*;\, t_0, \mathbf{x}_0)) < a_1(\eta) \tag{2.12}$$

Conversely, we assume that

$$V(t, \mathbf{y}_2(t;\, t_0, \mathbf{x}_0)) \geq a_1(\eta) \text{ при всех } t \in (t_0', t_0' + T_2(\eta))$$

Then, in this time interval $||\mathbf{y}_2(t; t_0, \mathbf{x}_0)|| \geq a_2^{-1}(a_1(\eta))$, and the relation (2.10) holds. This leads to the contradictory inequalities

$$0 < a_1(\eta) \leq V(t_0' + T_2(\eta), \mathbf{y}_2(t_0' + T_2(\eta); t_0, \mathbf{x}_0)) = V(t_0', \mathbf{y}_2(t_0'; t_0, \mathbf{x}_0)) +$$
$$+ \int_{t_0'}^{t_0' + T_2(\eta)} \dot{V}_{(1.2)}(\tau, \mathbf{y}_2(\tau; t_0, \mathbf{x}_0))d\tau \leq a_2(\eta) - \frac{1}{2}a_3(a_2^{-1}(a_1(\eta)))T_2(\eta) = \frac{1}{2}a_1(\eta)$$

From inequalities (2.11), (2.12) we conclude that

$$V(t, \mathbf{y}_2(t; t_0, \mathbf{x}_0)) < a_1(\eta) \text{ при всех } t \geq t_*$$

In fact, we assume, on the contrary that $V(t, \mathbf{y}_2(t; t_0, \mathbf{x}_0)) < a_1(\eta)$ for $t \in [t_*, t^*)$, but $V(t^*, \mathbf{y}_2(t^*; t_0, \mathbf{x}_0)) = a_1(\varepsilon)$. Then, we clearly have $\dot{V}_{(1.2)}(t^*, y_2(t^*; t_0, \mathbf{x}_0)) \geq 0$, which contradicts condition (2.12). Therefore, $||\mathbf{y}_2(t; t_0, \mathbf{x}_0)|| < \eta$ for $t \geq t_*$ on the basis of $V(t, \mathbf{y}_2) \geq a_1(||\mathbf{y}_2||)$. Consequently, $||\mathbf{y}_2(t; t_0, \mathbf{x}_0)|| < \eta$ for any $t \geq t_0 + T(\eta)$, where $T(\eta) = T_1(\eta) = T_2(\eta)$ if $||\mathbf{x}_0|| < \Delta < \delta^*$. The theorem is proved.

Supplement to Theorem 1. Condition *b* of Theorem 1 can be replaced by the following condition. In the region (1.3) a continuous scalar function $Y_2^*(\mathbf{y}_1, \mathbf{y}_2)$, $Y_2^*(0, \mathbf{y}_2) \equiv 0$ exists such that

$$|(\partial V(t, \mathbf{y}_2)/\partial \mathbf{y}_2)\mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z})| \leq |Y_2^*(\mathbf{y}_1, \mathbf{y}_2)| \tag{2.13}$$

where $V(t, \mathbf{y}_2)$ is the Lyapunov function that solves the uniform asymptotic stability problem of the equilibrium position $\mathbf{y}_2 = \mathbf{0}$ of the "reduced" subsystem (2.2).

Theorem 2. *Let conditions a and c of* Theorem 1 *hold, and in the region* (1.3) *let there be a vector function* $\mathbf{Y}_2^{**}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{z})$ *such that*

$$\|\mathbf{R}(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z})\| \leq \|\mathbf{Y}_2^{**}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{z})\|$$

Then

1) *if the equilibrium position* (1.4) *of system* (1.2) *is uniformly* $(\mathbf{y}_1, \mathbf{z})$*-stable and (simultaneously) uniformly asymptotically* $\mathbf{y}_1$*-stable, then it is uniformly* $\mathbf{x}$*-stable and (simultaneously) uniformly asymptotically* $\mathbf{y}_1$*-stable;*
2) *if the equilibrium position* (1.4) *of system* (1.2) *is uniformly* $(\mathbf{y}_1, \mathbf{z})$*-stable (uniformly asymptotically* $(\mathbf{y}_1, \mathbf{z})$*-stable), then it is uniformly* $\mathbf{x}$*-stable (uniformly asymptotically* $\mathbf{x}$*-stable).*

The proof follows the same scheme as the proof of Theorem 1.

Discussion of Theorems 1 and 2.

1°. Condition *b* of Theorem 1 is easily verified if the uniform $\mathbf{z}$-boundedness (with respect to t_0, $\mathbf{x}_0$) of the solutions of the system (1.2) that begin in a sufficiently small vicinity of the equilibrium position $\mathbf{x} = \mathbf{0}$ is known *a priori* from some arguments. Such a situation is typical of, for example, systems with a cylindrical phase space.[28] In the general case, a number of conditions for uniform boundedness of solutions of general classes of non-linear systems with respect to a part of the variables in the context of the Lyapunov function methodology can be found.[6] In stabilization problems with respect to a part of the variables, both uniform asymptotic stability with respect to a part of the variables and boundedness of the solutions of the closed system with respect to the remainder of the variables are often required (see Section 4).
2°. Condition *b*, in particular, holds if the function $\mathbf{R}$ does not depend on t, $\mathbf{z}$ (see Example 1) or is bounded with respect to t, $\mathbf{z}$.
3°. The conditions of Theorems 1 and 2 can be carried over to the class of mechanical systems of the form

$$\ddot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{X}(t, \mathbf{0}, \mathbf{0}) \in \mathbf{0}$$

For example, the conditions for uniform asymptotic stability of the equilibrium position $\mathbf{x} = \dot{\mathbf{x}} = \mathbf{0}$ with respect to the variables $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$, $\dot{\mathbf{y}} = (\dot{\mathbf{y}}_1^T, \dot{\mathbf{y}}_2^T)^T$ will include the condition for uniform asymptotic stability of this equilibrium position with respect to $\mathbf{y}_1, \dot{\mathbf{y}}_1$ and the condition for uniform asymptotic stability of the zero equilibrium position of the "reduced" subsystem with respect to the variables $\mathbf{y}_2, \dot{\mathbf{y}}_2$.

4°. Theorems 1 and 2 indicate fairly general classes of non-linear dynamical systems, which exhibit local partial zero-detectability ($\mathbf{y}_1/\mathbf{y}$ zero-detectability) with respect to the properties of uniform stability and uniform asymptotic stability. Other approaches to finding conditions for partial detectability, which are based either on the construction of Lyapunov functions with corresponding properties or on the use of differential-geometric methods, were proposed in Refs. 24–27.

5°. The proposed approach to solving the partial detectability problem and proving Theorems 1 and 2 relies on the ideas in Refs. 4, 6, 29. Confirmation of the second part of Theorem 2 was previously obtained in Ref. 4.

Example 1. Let system (1.2) consist of the equations

$$\dot{y}_1 = -ay_1 + e^{-t}y_2^2 z_1, \quad \dot{y}_2 = by_2 + y_1 y_2^2, \quad \dot{z}_1 = cz_1 - 2y_1 y_2 z_1 \tag{2.14}$$

where a, b and c are certain constants.

By introducing the new variable $\mu_1 = e^{-t}y_2^2 z_1$, from system (2.14) we can isolate the subsystem

$$\dot{y}_1 = -ay_1 + \mu_1, \quad \dot{\mu}_1 = (-1 + 2b + c)\mu_1$$

which defines the y_1-dynamics of system (2.14). Therefore, the equilibrium position

$$y_1 = y_2 = z_1 = 0 \tag{2.15}$$

of system (2.14) is uniformly y_1-stable for $a = 0$, $2b + c < 1$ and uniformly asymptotically y_1-stable for $a < 0$, $2b + c < 1$.

In this case, the "reduced" subsystem (2.2) reduces to the equation $\dot{y}_2 = by_2$, and when $b < 0$, the equilibrium position $y_2 = 0$ of this equation is uniformly asymptotically stable. If we also take into account that condition b of Theorem 1 holds for system (2.14), we can conclude that equilibrium position (2.15) of system (2.14) for $a = 0$, $2b + c < 1$ is not only uniformly y_1-stable, but also uniformly (y_1, y_2)-stable and that the equilibrium position for $a < 0$, $b < 0$, $2b + c < 1$ is not only uniformly asymptotically y_1-stable, but also uniformly asymptotically (y_1, y_2)-stable.

3. Application to non-linear controlled systems

Suppose system (1.2) describes the perturbed motion of a control object, taking into account the positional controls created by the design engineer. We will assume that the variables appearing in the vectors $\mathbf{y}_1$ and $\mathbf{z}$ are monitored by the design engineer and are used to create controls, and that the variables appearing in the vector $\mathbf{y}_2$ are not monitored. Suppose the controls created are such that the unperturbed motion (1.4) of system (1.2) is uniformly asymptotically $\mathbf{y}_1$-stable. Since the controls for $\mathbf{y}_1 = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$ are zero controls, the dynamics of subsystem (2.2) do not depend on the design engineer's controls and are determined solely by the structure and parameters of the object. We will assume that they were chosen so that the zero equilibrium position of subsystem (2.2) would be uniformly asymptotically stable. As a result, with this chosen structure and parameters of the object, uniform asymptotically $\mathbf{y}_1$-stability does, in fact, signify uniform asymptotic stability with respect to $\mathbf{y}_1$ and $\mathbf{y}_2$.

For the situation indicated in the second part of Theorem 2, this approach was previously considered as applied to the stability of the motion of an aircraft in Ref. 30.

4. Stabilization of an asymmetric rigid body with respect to an assigned direction in inertial space

Consider the dynamic Euler equations

$$A_1\dot{w}_1 = (A_2 - A_3)w_2w_3 + u_1 \ (123) \tag{4.1}$$

which describe the rotational motion of an asymmetric rigid body about its centre of mass (one equation is written out; the other two are obtained from it by cyclic permutation of the subscripts 1, 2 and 3). In system (4.1) w_i are the projections of the angular velocity vector of the body onto its principal central axes of inertia s_i, A_i are the principal central moments of inertia, u_i are the control moments, and $i = 1, 2, 3$.

Along with the equations defined by (4.1), let us consider the kinematic Poisson's equations that determine the orientation of the body

$$\dot{\gamma}_1 = w_3\gamma_2 - w_2\gamma_3 \ (123), \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{4.2}$$

in which γ_i are the projections of the unit vector directed along the vertical axis that is stationary in the inertial space onto the principal central axes.

When $u_i = 0$, Eqs. (4.1) and (4.2) allow of the particular solution

$$w_i = 0, \quad \gamma_1 = \gamma_3 = 0, \quad \gamma_2 = 1, \quad i = 1, 2, 3 \tag{4.3}$$

which corresponds to the equilibrium position of the body in which the direction of one of its principal central axes of inertia (the s_2 axis) coincides with the direction of the stationary vertical axis.

Introducing the new variables

$$y_{11} = \gamma_1, \quad y_{12} = \gamma_3, \quad y_{21} = w_1, \quad y_{22} = w_3, \quad z_1 = w_2$$

we compose the system of equations of the perturbed motion in deviations from the equilibrium position (4.3)

$$\begin{gathered}\dot{y}_{11} = -y_{12}z_1 + y_{22}f, \quad \dot{y}_{12} = -y_{21}f + y_{11}z_1 \\ A_1\dot{y}_{21} = (A_2 - A_3)y_{22}z_1 + u_1, A_3\dot{y}_{22} = (A_1 - A_2)y_{21}z_1 + u_3, A_2\dot{z}_1 = (A_3 - A_1)y_{21}y_{22} + u_2 (4.4) \\ f = (1 - y_{11}^2 - y_{12}^2)^{1/2}\end{gathered} \tag{4.4}$$

We introduce the notation

$$\mathbf{x} = (\mathbf{y}^T, \mathbf{z}_1)^T, \quad \mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T, \quad \mathbf{y}_1 = (y_{11}, y_{12})^T, \quad \mathbf{y}_2 = (y_{21}, y_{22})^T$$

Problem 2. Find the control moments u_i that solve the partial stabilization problem of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (4.4): the **y**-stabilization problem of this equilibrium position. Here the behaviour of the closed system (4.4) with respect to the variable z_1 must be specified by the relation

$$z_1 = \pm\omega = \text{const}, \quad t \to \infty \tag{4.5}$$

where ω is an *a priori* assigned number.

This partial stabilization problem calls for stabilization of one of the principal central axes of inertia (the s_2 axis) in the direction of the stationary vertical axis. When solving the problem, unlike the stabilization of the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (4.4) with respect to all the variables, the angular velocity of the body about the s_2 axis does not vanish. Instead, the magnitude of this angular velocity tends to the value ω that was assigned *a priori* (the direction of rotation plays no role).

Assertion 1. The solution of Problem 2 gives the non-linear control moments

$$
\begin{aligned}
u_1 &= A_1 f^{-1}[y_{11}y_{21}y_{22} - y_{12}(y_{21}^2 + z_1^2) + y_{11}z_1(-z_1^2 + \omega^2) + u_1^*] + A^* y_{22}z_1 \\
u_2 &= A_2 z_1(-z_1^2 + \omega^2) + (A_1 - A_3)y_{21}y_{22} \\
u_3 &= A_3 f^{-1}[-y_{12}y_{21}y_{22} + y_{11}(y_{22}^2 + z_1^2) + y_{12}z_1(-z_1^2 + \omega^2) + u_2^*] - A^* y_{21}z_1 \\
u_1^* &= -\lambda_1 y_{12} - \lambda_2(-y_{21}f + y_{11}z_1), \quad u_2^* = \lambda_3 y_{11} + \lambda_4(-y_{12}z_1 + y_{22}f) \\
\lambda_j &= \text{const} < 0, \quad j = 1, 2, 3, 4, \quad A^* = (A_1 - A_2 + A_3)
\end{aligned}
\tag{4.6}
$$

Proof. Omitting the intermediate mathematical operations, it can be shown that closed system (4.4), (4.6) yields the linear system of differential equations

$$\ddot{y}_1 = \lambda_3 y_{11} + \lambda_4 \dot{y}_{11}, \quad \ddot{y}_{12} = \lambda_1 y_{12} + \lambda_2 \dot{y}_{12} \tag{4.7}$$

Since the $\mathbf{y}_1$-dynamics of closed system (4.4), (4.6) are defined by system (4.7), the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (4.4), (4.6) is uniformly asymptotically $\mathbf{y}_1$-stable; $\mathbf{y}_1 = (\mathbf{y}_{11}, \mathbf{y}_{12})^T$. The "reduced" subsystem of the type (2.2) consists of the equations

$$\dot{y}_{21} = \lambda_2 y_{21}, \quad \dot{y}_{22} = \lambda_4 y_{22} \tag{4.8}$$

and its zero equilibrium position $\mathbf{y}_2 = (\mathbf{y}_{11}, \mathbf{y}_{12})^T = \mathbf{0}$ is uniformly asymptotically Lyapunov stable.

The components R_i of the vector $\mathbf{R}$-function have the form

$$R_i(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}) = \Lambda_i(\mathbf{y}_2, \mathbf{z}) + R_i^*(t, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}), \quad R_i^*(t, \mathbf{0}, \mathbf{y}_2, \mathbf{z}) \equiv 0, \quad i = 1, 2$$

$$\Lambda_1 = y_{22}z_1, \quad \Lambda_2 = -y_{21}z_1$$

and the z_1 variable of system (4.4), (4.6) is bounded. Therefore, condition (2.13) holds for the Lyapunov function $V = y_{21}^2 + y_{22}^2$ that solves the uniform asymptotic stability problem of the "reduced" subsystem (4.8).

Based on the supplement to Theorem 1, we conclude that the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (4.4), (4.6) is not only uniformly asymptotically $\mathbf{y}_1$-stable, but also uniformly asymptotically $\mathbf{y}$-stable; $\boldsymbol{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$. The dynamics of the z_1 variable of system (4.4), (4.6) are specified by the equation $\dot{z}_1 = z_1(-z_1^2 + \omega^2)$, and, therefore, the limiting relation (4.5) holds. The assertion is proved.

Remarks.

1°. Problem 2 is the problem of stabilizing the equilibrium position $\mathbf{x} = \mathbf{0}$ of system (4.4) with respect to a part of the variables by means of additional control moments. We note that $u_i(\mathbf{x}) \to 0$ when $t \to \infty$ ($i = 1, 2, 3$).
2°. The proposed approach to solving Problem 2 can be extended to the case of control by means of flywheels.

5. The conditions for partial detectability of linear systems with constant coefficients

For linear systems with constant coefficients, along with the problem of stability with respect to a part of the variables,[10,12,15] the problem of finding a solution with respect to a part of the variables,[31] whose analysis by classical computational methods[32,33] is difficult, was also considered. In this context it would be interesting to analyse the previously untreated partial detectability problem of linear systems, which can be used to find the conditions under which asymptotic stability of the system with respect to one part of the variables signifies its asymptotic stability with respect to the other, larger part of the variables. These conditions call for an analysis of only the structural forms of the system without an analysis of their asymptotic stability with respect to the respective groups of variables, and after appropriate refinement of the notion of partial detectability, these conditions are not only sufficient, but also necessary. Unlike non-linear systems, in which the Lyapunov function methodology is used to analyse partial detectability, the analysis of linear systems relies on the mathematical tools of linear algebra.

Let (1.1) be a linear system of ordinary differential equations with constant coefficients. Taking into account the splitting of the **x**-vector into three parts described in Section 1, we present this system in the form of three groups of equations

$$\dot{\mathbf{y}}_i = A_i\mathbf{y}_1 + B_i\mathbf{y}_2 + C_i\mathbf{z}, \quad i = 1, 2, \quad \dot{\mathbf{z}} = A_3\mathbf{y}_1 + B_3\mathbf{y}_2 + C_3\mathbf{z}$$
$$\mathbf{y}_1 \in R^m, \quad \mathbf{y}_2 \in R^k, \quad \mathbf{z} \in R^p \tag{5.1}$$

where A_i, B_i, C_i ($i=1, 2, 3$) are constant matrices of the corresponding dimensions. We also set $\mathbf{y} = (\mathbf{y}_1^T, \mathbf{y}_2^T)^T$.

The linear system (5.1) is asymptotically $\mathbf{y}_1$-stable (asymptotically **y**-stable) if the $\mathbf{y}_1$-component (**y**-component) of the solution $\mathbf{x}(t)$ tends to zero as $t \to \infty$ and for all $t_0 \geq 0$, $\mathbf{x}_0$.

We stipulate that the $\mathbf{y}_1$- and $\mathbf{y}_2$-components of the solutions of system (5.1) have an identical structure if the dimensions of the system[10,12] that specify the $\mathbf{y}_1$- and $\mathbf{y}_2$-dynamics of system (5.1) are identical and the same as the dimensions of the system that specifies its **y**-dynamics.

Definitions. System (5.1) is called:

1) *partially detectable* ($\mathbf{y}_1/\mathbf{y}$-detectable) if the asymptotic $\mathbf{y}_1$-stability of this system signifies its asymptotic **y**-stability;
2) *strongly* $\mathbf{y}_1/\mathbf{y}$-*detectable* if it is $\mathbf{y}_1/\mathbf{y}$-detectable and, in addition, the $\mathbf{y}_1$- and $\mathbf{y}_2$-components of the solutions have an identical structure.

Let us consider the problems of the partial and strong partial detectability of linear system (5.1). We introduce the matrices

$$K_1 = \|D^T, G^TD^T, \dots, (G^T)^{k+p-1}D^T\|, \quad K_2 = \|L^T, C_3^TL^T, \dots, (C_3^T)^{p-1}L^T\|$$

$$D = \|B_1, C_1\|, \quad G = \begin{Vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{Vmatrix}, \quad L = \begin{Vmatrix} C_1 \\ C_2 \end{Vmatrix}$$

Theorem 3. *If the condition*

$$\operatorname{rank}K_1 = k + \operatorname{rank}K_2 \tag{5.2}$$

holds, system (5.1) *is* $\mathbf{y}_1/\mathbf{y}$-*detectable*.

Proof. If condition (5.2) holds, the $\mathbf{y}_1$-dynamics and **y**-dynamics of system (5.1) will be specified[10,12] by linear auxiliary systems of the same dimension (the dimension $m+\operatorname{rank} K_1 = m+k+\operatorname{rank} K_2$) and this dimension will not exceed the dimension of system (5.1). Since the first m components of the solutions of these auxiliary systems of the same dimension are identical, the sets of roots of the characteristic equation of these auxiliary systems are subsets of the set of roots of the characteristic equation of system (5.1) and are also identical. In such a situation, asymptotic $\mathbf{y}_1$-stability of system (5.1) will signify its asymptotic **y**-stability. The theorem is proved.

Discussion of Theorem 3.

1°. The condition (5.2) for partial detectability calls for an analysis *of only the structural form* of system (5.1) without an analysis of its asymptotic stability with respect to the corresponding groups of variables. In this sense, we can understand why condition (5.2) does not cover the cases of "weak" coupling in system (5.1), for example, the case in which B_1 and C_1 are zero matrices.
2°. The condition rank $K_1 = k+p$ for the detectability ($\mathbf{y}_1/\mathbf{x}$-detectability) of system (5.1), which was previously obtained in Refs. 10,12 when solving problems of stability with respect to a part of the variables, follows from condition (5.2) as a special case. (Note that the term "detectability" was not used in this case.)
3°. In the case in which system (5.1) is unstable or neutral with respect to the **z**-variables, the partial detectability (as well as partial stability) can be disrupted already for a small variation of its coefficients (in this context, see Refs. 10,12,15).

To study strong partial detectability we consider the matrices

$$K_3 = \left\| M^T, N^T M^T, \ldots, (N^T)^{m+p-1} M^T \right\|, \quad M = \|A_2, C_2\|, \quad N = \left\| \begin{matrix} A_1 & C_1 \\ A_3 & C_3 \end{matrix} \right\|$$

Theorem 4. For strong $\mathbf{y}_1/\mathbf{y}$-detectability of system (5.1) it is necessary and sufficient that the condition

$$m + \operatorname{rank} K_1 = m + k + \operatorname{rank} K_2 = k + \operatorname{rank} K_3 \tag{5.3}$$

holds.

Proof. *Necessity.* Let the system be strongly $\mathbf{y}_1/\mathbf{y}$-detectable. In this case, the dimensions of the systems that specify the $\mathbf{y}_1$- and $\mathbf{y}_2$-dynamics of system (5.1) are identical and are the same as the dimension of the system that specifies its $\mathbf{y}$-dynamics. According to the results previously obtained,[10,12] the equalities in (5.3) hold in this case.

Sufficiency. Let the equalities in (5.3) hold. In this case, the dimensions of the systems that specify the $\mathbf{y}_1$- and $\mathbf{y}_2$-dynamics of system (5.1) are identical and are the same as the dimension of the system that specifies its $\mathbf{y}$-dynamics.[10,12] In such a situation, asymptotic $\mathbf{y}_1$-stability of system (5.1) will signify its asymptotic $\mathbf{y}_2$-stability. As a result, system (5.1) is strongly $\mathbf{y}_1/\mathbf{y}$-detectable.

Example 3. Let system (5.1) have the form

$$\begin{aligned} \dot{y}_1 &= -2y_1 + y_2 + z_1 - 2z_2, \quad \dot{y}_2 = y_1 - y_2 - \varepsilon(z_1 - 2z_2) \\ \dot{z}_1 &= 4y_1 + 2y_2 + z_1, \quad \dot{z}_2 = 2y_1 + y_2 + z_1 - z_2 \end{aligned} \tag{5.4}$$

where ε is some constant. In this case

$$k = 1, \quad p = 2, \quad K_1 = \left\| \begin{matrix} 1 & -1 & 1 \\ 1 & -(1+\varepsilon) & 1+2\varepsilon \\ -2 & 2(1+\varepsilon) & -2(1+2\varepsilon) \end{matrix} \right\|, \quad K_2 = \left\| \begin{matrix} 1 & -\varepsilon & -1 & \varepsilon \\ -2 & 2\varepsilon & 2 & -2\varepsilon \end{matrix} \right\|$$

$$K_3 = \left\| \begin{matrix} 1 & -2 & 4 \\ -\varepsilon & 1+\varepsilon & -(3+\varepsilon) \\ 2\varepsilon & -2(1+\varepsilon) & 2(3+\varepsilon) \end{matrix} \right\|$$

$$\operatorname{rank} K_1 = k + \operatorname{rank} K_2 = 2 < k + p = 3 \text{ при } \varepsilon \neq 0$$

$$\operatorname{rank} K_1 = 1 < k + \operatorname{rank} K_2 = 2 < k + p = 3 \text{ при } \varepsilon = 0$$

$$m + \operatorname{rank} K_1 = m + k + \operatorname{rank} K_2 = k + \operatorname{rank} K_3 = 3 < n = 4 \text{ при } \varepsilon \neq 0$$

and condition (5.2) holds for all $\varepsilon \neq 0$. Therefore, on the basis of Theorem 3, when $\varepsilon \neq 0$, system (5.4) is $y_1/\mathbf{y}$-detectable, and $\mathbf{y} = (y_1, y_2)$. In addition, when $\varepsilon \neq 0$, condition (5.3) holds, and on the basis of Theorem 4, strong $y_1/\mathbf{y}$-detectability of the system (5.4) occurs.

In the case when $\varepsilon = 0$, although condition (5.2) does not hold, system (5.4) is also $y_1/\mathbf{y}$-detectable, but unlike the case when $\varepsilon \neq 0$, the dimension of the auxiliary system

$$\dot{y}_1 = -2y_1 + \mu_1, \quad \dot{\mu}_1 = y_1 - \mu_1$$

which describes the behaviour of the variable y_1, is smaller than the dimension of the auxiliary system

$$\dot{y}_1 = -2y_1 + y_2 + \mu_2, \quad \dot{y}_2 = y_1 - y_2, \quad \dot{\mu}_2 = -\mu_2$$

which describes the behaviour of the variables y_1, y_2.

Therefore, when $\varepsilon = 0$, system (5.4) is not strongly $y_1/\mathbf{y}$-detectable. Condition (5.3) also does not hold in this case.

6. Conclusions

New conditions for partial zero-detectability have been obtained for non-linear non-stationary dynamical systems of a general type. Unlike the previous studies,[24–27] in which the conditions for partial zero-detectability are formulated in the context of not always easily tested requirements for the Lyapunov functions or they require fairly complex transformations of the original system, the proposed approach provides a way to obtain easily interpreted conditions for partial detectability based on an analysis of the structural forms of systems that can be studied directly. The Lyapunov function methodology is used here only as a means for obtaining such conditions. For linear systems with constant coefficients, conditions for partial detectability that call for an analysis of only the structural forms of these systems have been obtained. The classical conditions for complete (not partial) detectability of linear systems follow from these conditions.

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References

1. Lyapunov AM. Investigation of a special case of the problem of the stability of motion. In: Lyapunov AM, editor. *Collected Papers*, 2. Moscow, Leningrad: Izd Akad Nauk SSSR; 1956. p. 272–331.
2. Rumyantsev VV. The stability of motion with respect to a part of the variables. *Vestn Mosk Univ Ser Mat Mekh Astron Fiz Khim* 1957;**4**:9–16.
3. Corduneanu S. Sur la stabilité partielle. *Rev Roum Math Pure Appl* 1964;**9**(3):229–36.
4. Halanay A. *Differential Equations: Stability, Oscillations, Time Lags*. New York: Academic Press; 1966.
5. Pieffer K, Rouche N. Liapounov's second method applied to partial stability. *J Mech* 1969;**8**(2):323–34.
6. Rumyantsev VV, Oziraner AS. *Stability and Stabilization of Motion with Respect to a Part of the Variables*. Moscow: Nauka; 1987.
7. Savchenko AYa, Ignat'yev AO. *Some Stability Problems of Non-Autonomous Systems*. Kiev: Naukova Dumka; 1989.
8. Hatvani L. On the stability of the solutions of ordinary differential equations with mechanical applications. *Alkalm Mat Lap* 1990/1991;**1–2**:1–90.
9. Andreyev AS. Investigation of partial asymptotic stability. *Prikl Mat Mekh* 1991;**55**(4):539–47.
10. Vorotnikov VI. *Stability of Dynamical Systems with Respect to a Part of the Variables*. Moscow: Nauka; 1991.
11. Khapaev MM. *Averaging in Stability Theory*. Dordrecht: Kluwer; 1993.
12. Vorotnikov VI. *Partial Stability and Control*. Boston: Birkhauser; 1998.
13. Fradkov AL, Miroshnik IV, Nikiforov VO. *Nonlinear and Adaptive Control of Complex Systems*. Dordrecht: Kluwer; 1999.
14. Vorotnikov VI. Problems of stability with respect to a part of the variables. *Prikl Mat Mekh* 1999;**63**(5):736–45.
15. Vorotnikov VI, Rumyantsev VV. *Stability and Control with Respect to a Part of the Coordinates of the Phase Vector of Dynamical Systems: Theory, Methods and Applications*. Moscow: Nauch Mir; 2001.
16. Chellaboina V, Haddad WM. A unification between partial stability and stability theory for time-varying systems. *IEEE Control Syst Magazine* 2002;**22**(6):66–75 (Erratum. *IEEE Control Syst Magazine* 2003;**23**(1):103).
17. Michel AN, Molchanov AP, Sun Y. Partial stability and boundedness of general dynamical systems on metric spaces. *Nonlinear Anal TMA* 2003;**52**(4):1295–316.
18. Jian JG, Liao XX. Partial exponential stability of nonlinear time-varying large-scale systems. *Nonlinear Anal TMA* 2004;**59**(5):789–800.
19. Zuyev AL. Partial asymptotic stability of abstract differential equations. *Ukr Mat Zh* 2006;**58**(1):629–37.
20. Vorotnikov VI. Partial stability and control: state of the problem and prospects for development. *Avtomat Telemekh* 2006;**4**:3–59.
21. Sontag ED, Wang Y. Output-to-state stability and detectability of nonlinear systems. *Syst Control Lett* 1997;**29**(4–5):279–90.
22. Sontag ED. *Mathematical Control Theory; Deterministic Finite Dimensional Systems*. New York: Springer; 1998.
23. Sontag ED. Input to state stability: Basic concepts and results. In: *Nonlinear and Optimal Control Theory*. Berlin: Springer; 2006.
24. Shiriaev AS, Fradkov AL. Stabilization of invariant sets for nonlinear non-affine systems. *Automatica* 2000;**36**(11):163–220.
25. Shiriaev AS. The notion of V-detectability and stabilization of invariant sets of nonlinear systems. *Syst Control Lett* 2000;**39**(5):327–38.
26. Ingalls BP, Sontag ED, Wang Y. Measurement to error stability: a notion of partial detectability for nonlinear systems. In: *Proceedings of the 2002 IEEE Conference on Decision and Control, Las Vegas, Nevada*. 2002. p. 3946–51.
27. Yefimov DV. *Robust and Adaptive Control of Nonlinear Oscillations*. St. Petersburg: Nauka; 2005.
28. Barbashin YeA, Tabuyeva VA. *Dynamical Systems with a Cylindrical Phase Space*. Moscow: Nauka; 1969.
29. Oziraner AS. A Malkin–Masser theorem. *Prikl Mat Mekh* 1979;**43**(6):975–9.
30. Hacker T. Stability of partially controlled motion of an aircraft. *J Aerospace Sci* 1961;**28**(1):15–26.
31. Abramov SA, Bronshtein M. Solution of linear differential and difference systems with respect to a part of the unknowns. *Zh Vichisl Mat Mat Fiz* 2006;**46**(2):229–41.
32. Bronstein M. Computer algebra algorithms for linear ordinary differential and difference equations. *Prog Math* 2001;**202**:105–19.
33. van der Put M, Singer MF. *Galois Theory of Linear Differential Equations*. Heidelberg: Springer; 2003.

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